SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 10

MATH 241

19.5.4

Proof. $f(z) = (z+3)^2 \sin \frac{2}{z+3} = (z+3)^2(\frac{2}{z+3} - \frac{1}{3!}(\frac{2}{z+3})^3 + \cdots) = 2(z+3) - \frac{1}{6}\frac{8}{z+3} + \cdots$, so the coefficient of the degree -1 term is $-\frac{1}{6} \cdot 8 = -\frac{4}{3} = Res(f(z), -3)$. □

19.5.15

Proof. $\sec z = \frac{1}{\cos z}$, so the poles are $z = \frac{\pi}{2} + n\pi$. $\lim_{z \to \frac{\pi}{2}+n\pi} (z - (\frac{\pi}{2} + n\pi)) \sec z = \lim_{z \to \frac{\pi}{2}+n\pi} \frac{z-(\frac{\pi}{2}+n\pi)}{\cos z}$, by L'hospital rule this limit is equal to $\lim_{z \to \frac{\pi}{2}+n\pi} \frac{1}{-\sin z} = \pm 1$ depending on n odd or even. So the poles $\frac{\pi}{2} + n\pi$ are all simple poles, the residue is 1 if n is odd, -1 if n is even. □

19.5.22

Proof. 1 is the only pole inside the contour, so $\oint_C \frac{1}{z^3(z-1)^4} dz = 2\pi i Res(f(z), 1)$. Since 1 is a pole of order 4, let $g(z) := (z-1)^4 f(z) = \frac{1}{z^3}$, $g^{(3)}(z) = (-3)(-4)(-5)z^{-6} = -60z^{-6}$, so $g^{(3)}(1) = -60$. Therefore $Res(f(z), 1) = \frac{1}{3!} g^{(3)}(1) = -10$. So $\oint_C \frac{1}{z^3(z-1)^4} dz = 2\pi i Res(f(z), 1) = -20\pi i$. □

19.6.9

Proof. Let $z = e^{i\theta}$, so $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z+\frac{1}{z}}{2}$. What about $\cos 2\theta$? Since $z^2 = e^{2i\theta} = \cos(2\theta) + i\sin(2\theta)$, so $\cos(2\theta) = \frac{z^2+z^{-2}}{2}$. Therefore the integral becomes

$$\oint_C \frac{\frac{z^2+z^{-2}}{2}}{5 - 4 \cdot \frac{z+\frac{1}{z}}{2}} \frac{dz}{iz} = -\frac{1}{2i} \oint_C \frac{z^4+1}{z^2(2z^2-5z+2)} dz$$

Where C is the unit circle oriented counterclockwisely. Let $f(z) := \frac{z^4+1}{z^2(2z^2-5z+2)}$. The poles are $0, 2, \frac{1}{2}$. 0 and $\frac{1}{2}$ are inside the contour. So the answer is equal to $-\frac{1}{2i} 2\pi i \cdot (Res(f(z), 0) + Res(f(z), \frac{1}{2})) = -\pi(Res(f(z), 0) + Res(f(z), \frac{1}{2}))$.

0 is a pole of order 2, so let $g(z) = z^2 f(z) = \frac{z^4+1}{2z^2-5z+2}$, and $g'(z) = \frac{4z^3(2z^2-5z+2)-(z^4+1)(4z-5)}{(2z^2-5z+2)^2}$, plug in $z = 0$ you will get $\frac{5}{4} = Res(f(z), 0)$.

$\frac{1}{2}$ is a simple pole, so if we let $h(z) := z^2(2z^2 - 5z + 2)$ to be the denominator, $h'(z) = 2z(2z^2 - 5z + 2) + z^2(4z - 5)$, plug in $z = \frac{1}{2}$ one get $-\frac{3}{4}$. So $Res(f(z), \frac{1}{2}) = \frac{(\frac{1}{2})^4+1}{-\frac{3}{4}} = -\frac{17}{12}$.

Therefore the answer is $-\pi(Res(f(z), 0) + Res(f(z), \frac{1}{2})) = -\pi(\frac{5}{4} - \frac{17}{12}) = \frac{\pi}{6}$. □

Fall 11, #8

Proof. $I_1 = I_2$ by Cauchy-Goursat Theorem since 0 is the only singularity of the integrand, and the deformation from γ_1 to γ_2 does not pass 0. Their value is equal to $2\pi i Res(z^3 e^{-\frac{1}{z^2}}, 0)$. The Laurent expansion at a punctured disk around 0 is $z^3(1 - \frac{1}{z^2} + \frac{1}{2!}(\frac{1}{z^2})^2 - \cdots) = z^3 - z + \frac{1}{2}\frac{1}{z} - \cdots$, so $Res(z^3 e^{-\frac{1}{z^2}}, 0) = \frac{1}{2}$. Therefore $I_1 = I_2 = \pi i$. $I_3 = 0$ because the integrand is analytic inside γ_3. Therefore $I_1 = I_2 \neq I_3$. □